Global Convergence of a Charge Pump PLL using Lyapunov Stability and Reachability

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Outline

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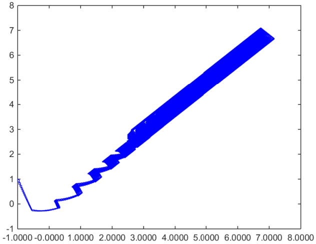
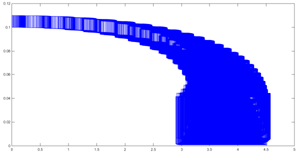
- Objective of the Research
- Lyapunov Analysis
- Set Advection
- Results
- Conclusion and Future Work

Objective of the Research

- Global Convergence of CP PLL is an important property.
- Hybrid System Modelling Paradigm.
- Use Reachability Computation
  - State space is divided in to patricians.
  - High Granularity is required.
  - Convergence to lock up condition is verified in bounded time.
- Several issues to this approach.
  - CP PLL needs hundreds of discrete transitions.
  - High number of reach set computations in additions to intersection with Guards.
  - Needs large Memory and computation resources.
  - Tools (StateEx,Flow) timed out during reachability computations.
- Proposed Solutions
  - Lyapunov Stability+Reachability.



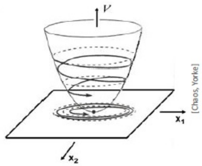


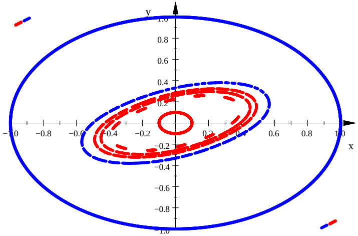
Generated by the tool Flow*

http://systems.cs.colorado.edu/research/cyberphysical/taylormodels/

Lyapunov Stability

- $\dot{x} = f(x) \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n$
- Lyapunov analysis use an abstract energy like function proving Stability (Asymptotic Stability) .
- A function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\dot{V}(x) = \langle \nabla V, f \rangle$
- $V(x) > 0$ and $-\dot{V}(x) > 0 \implies$ GAS
- Level sets described by the level curves of V are Invariant sets.





Invariant set around the equilibrium for CP PLL Hybrid System using Lyapunov Stability

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- Hybrid system $\dot{x} = f_\ell(x), \quad \ell \in L = \{1, ..., N\}$
- $\mathcal{X}_\ell = \{x \in \mathbb{R}^n : g_{\ell k} \geq 0, \text{ for } k = 1, ..., m_{\mathcal{X}_\ell} \text{ where } g_{\ell k} : \mathbb{R}^n \to \mathbb{R}\}$.
- $G(\ell, \ell') = \{x \in \mathbb{R}^n : h_{\ell\ell' 0}(x) = 0, \ h_{\ell\ell' k}(x) \geq 0, \text{ for } k = 1, ..., m_{\mathcal{X}_\ell} \text{ where } h_{ijk} : \mathbb{R}^n \to \mathbb{R}\}$.
- $R(\ell, \ell')(x) = \psi_{\ell\ell'}(x)$
- Global Lyapunov Certificate: If $R(\ell, \ell')(x) = x$ and open set $\mathcal{S} \subset \mathbb{R}^n$ such that $0 \in \mathcal{S}$ Let $V : \mathcal{S} \to \mathbb{R}$ be a continuously function such that
  - $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathcal{S} \setminus \{0\}$,
  - $\langle \nabla V, f_\ell \rangle \leq 0$ for all $x \in \mathcal{S}, \ \ell \in L$
- x=0 is stable. If $\langle \nabla V, f_\ell \rangle < 0$, then AS.
- Such a global Lyapunov certificate is difficult to construct.
- Use Multiple Lyapunov Certificates Instead for each mode,
  - $V_\ell(0) = 0$ and $V_\ell(x) > 0$ for all $x \in \mathcal{X}_\ell \setminus \{0\}$,
  - $\langle \nabla V_\ell, f_\ell \rangle \leq 0$ for all $x \in \mathcal{X}_\ell, \ \ell \in L$
  - $V'_\ell(x') \leq V_\ell(x)$ for all $x \in G(\ell, \ell'), \ x' = R(\ell, \ell')(x)$
- Invariant set $\bigcup_\ell V_\ell \leq c$ for all $\ell \in L$ if $\bigcap_\ell \mathcal{X}_\ell = \emptyset$

Antonis Papachristodoulou et al. Robust Stability Analysis of Nonlinear Hybrid Systems.IEEE Transaction on Automatic Control 2009

- Constructing Lyapunov certificate involves positivity test of $V(x)$ and $-\langle \nabla V_\ell, f_\ell \rangle$
- Checking Positivity an NP-hard problem.
- Sufficient condition for $p(x)$, $p(x) = \sum_{i=1}^{m} p_i^2(x)$ i.e. SOS decomposition.
- In Gram matrix form as $p(x) = Z^T(x)QZ(x)$, where $Z(x)$ is a vector of monomials and Q is a positive semi-definite matrix.
- Positivity check Boils down to the search for a positive semi-definite matrix Q and semi-definite programming can be used for its construction.

Constructing Multiple Lyapunov functions using SOS Programming

- We convert the Lyapunov stability conditions as SOS constraints

$$V_\ell(x) - \sum_{k=1}^{m_{\mathcal{X}_\ell}} s1_{\ell k}(x) g_{\ell k}(x) \text{ is SOS}$$

$$-\langle \nabla V_\ell, f_\ell \rangle - \sum_{k=1}^{m_{\mathcal{X}_\ell}} s2_{\ell k}(x) g_{\ell k}(x) \text{ is SOS}$$

$$V_\ell(x) - V'_\ell(x') - \sum_{k=1}^{m_{\mathcal{X}_\ell}} s3_{\ell\ell' k}(x) h_{\ell\ell' k}(x) -$$

$$s4_{\ell\ell' 0}(x, x') h_{\ell\ell' 0}(x) - s5_{\ell\ell'}(x' - \psi_{\ell\ell'}(x)) \text{ is SOS } \forall \ell, \ell'.$$

- Here $s1_{\ell k}, s2_{\ell k},$ *and* $s3_{\ell\ell' k}$ are all SOS polynomials.
- Union/Intersection of the level sets of V_ℓ is an invariant set.

Level sets Intersection/Union

- Let $p, q \in \mathbb{R}[x]$, $\mathbb{R}[x]$ is ring of polynomials in x with real coefficients.
- Exists two sum of square polynomials, s_0, s_1,

$$s_0 - s1q + p = 0 \ \ \forall x \in \mathbb{R}^n$$

Then Zero-Sub-level(q) $\subset$ Zero-Sub-level(p).

- Given q and the degree bound of p, s_0, s_1, the set of coefficients of p, s_0, s_1 satisfying the above constraint is the feasible set of a semi-definite program.
- Using this lemma to find sets union/intersection of the candidate Lyapunov functions.
- if $V - \gamma \leq 0$ is the desired zero level set, then we use an optimization using SOS programming,

maximize γ
subject to $s_0 + s1p + \epsilon - (V - \gamma) = 0,$

s_0, s_1 are SOS Polynomials, and $p \leq 0$ is the region of our interest.

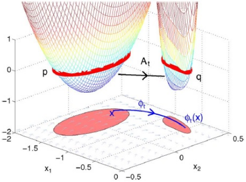
Ta-Chung Wang et al. Polynomial level-set method for attractor estimation.Journal of the Franklin Institute 2012

Set Advection to enlarge the Invariant Region around the equilibrium

- Reachability computations can be used to show convergence to the initial optimized invariant set.
- Instead, we use Set Advection to show that the Lyapunov invariant region is reachable from all states.
- In its simple form,

  $q = A_t p$ if $\quad q(x) = p(\phi_t(x)) \; \forall x \in \mathbb{R}^n$

- If $q = A_t p$, then
  $Zero - Sub - level(q) = \phi_t(Zero - Sub - level(p))$
- In its simple form
  $s1 - s2q + B_{h-\alpha} p = 0$
  $s3 + s4q - B_h p = 0$
  B_h is an approximation to the advection operator.



Advection Operator

Ta-Chung Wang et al. Polynomial level-set method for attractor estimation. Journal of the Franklin Institute 2012

Charge Pump Phase Lock Loop as a Hybrid System

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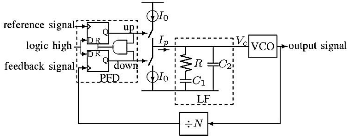
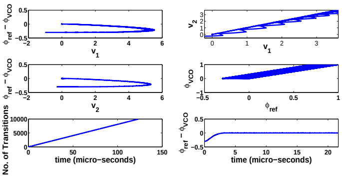
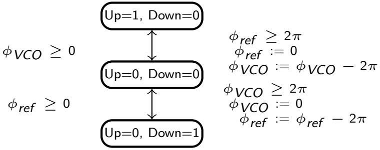


Charge Pump PLL



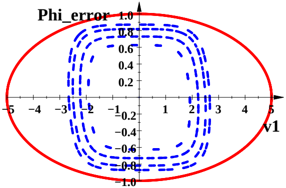
Simulations Plots of the CP PLL Hybrid System

$$I_p = \begin{cases} \in [I_P^U \; I_P^U] & \text{UP=1, Down=0, } (0 \leq \phi_{VCO} < 2\pi \leq \phi_{ref}) \\ \in [I_P^D \; I_P^D] & \text{UP=0, Down=1, } (0 \leq \phi_{ref} < 2\pi \leq \phi_{VCO}) \\ \in [0^R \; 0^R] & \text{UP=0, Down=0, } (0 \leq \phi_{VCO}, \phi_{ref} < 2\pi) \end{cases}$$

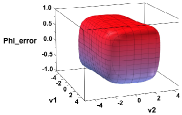


CP PLL Hybrid System

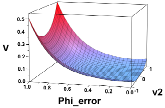
- ϕ_{ref}, $and \phi_{VCO}$ do not converge to zero.
- We consider $\phi_{error} = \phi_{ref} - \phi_{VCO}$ as an abstract state variable.
- Show stability of the equilibrium $\phi_{error} = 0$, $v1(\text{Voltage across} C1) = 0$, $v2(\text{Voltage across} C2) = 0$



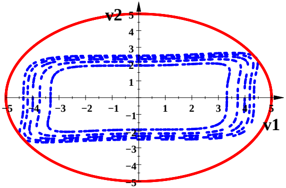
Level Curves of Lyapunov function (Degree 6) for Mode (Up=0, Down=0) Projected on v1-ϕ_{error}



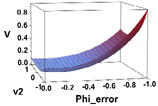
Lyapunov function for Mode (Up=0, Down=0) in 3D



Lyapunov function for Mode (Up=1, Down=0) in 2D



Level Curves of Lyapunov function (Degree 6) for Mode (Up=0, Down=0) Projected on v1-v2



Lyapunov function for Mode (Up=0, Down=1) in 2D

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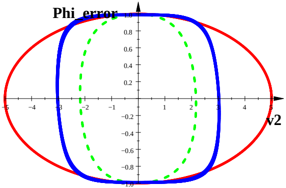
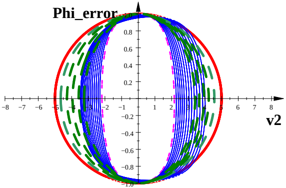


Initial Invariant Set Projected on v1-ϕ_{error}



Global Convergence Using Stability and Reachability
Red := State Space
Magenta := Initial Invariant Set
Blue := Backward Advection
Green := Forward Advection

- Time Taken by Multiple Lyapunov Computations = 862.0147 Seconds
- Time Taken by Intersection of functions = 120.3860 Seconds
- Time Taken by maximizing the Initial Invariant Set = 5.4132 Seconds
- Time Taken by forward Advection of sets = 350 Seconds (Approximately)
- Time Taken by Backward Advection of sets = 30 Seconds (Approximately)

- We have shown a solution to the problem of using only reachability for Global Convergence of the CP PLL.
- Results shows that Lyapunov based analysis has a great potential in AMS circuit verification.
- Though needs quite a bit of human interaction, SOS programming offers solutions to a range of problems .
- Robust Stability analysis can be done introducing additional inequalities and SOS multipliers.
- We are aiming to try and increase as much as possible the initial invariant set.
- We aim to extend the methodology to the safety verification of other complex circuits.

THANKS